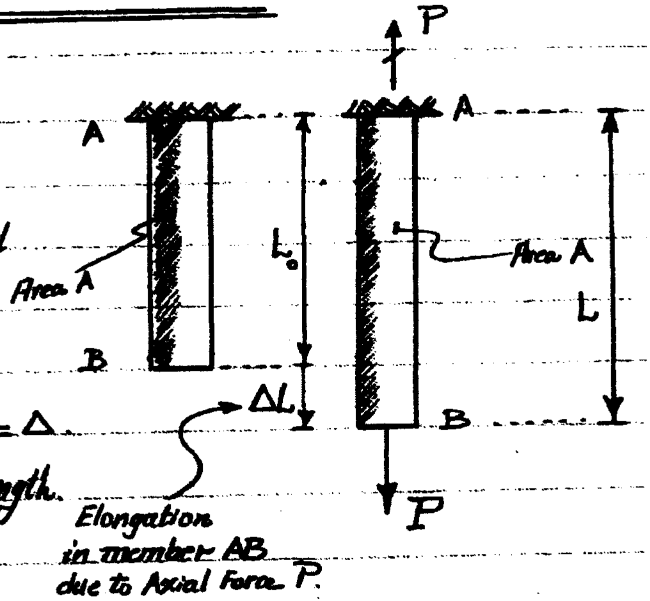
Axial Deformation Of Bars

Axial Strain

Consider a bar AB of cross-sectional Area A and length L_o, subjected to axial load P.

⇒ Axial deformation (elongation) = Δ.

$\Delta L = (L - L_o)$ .... change in length.

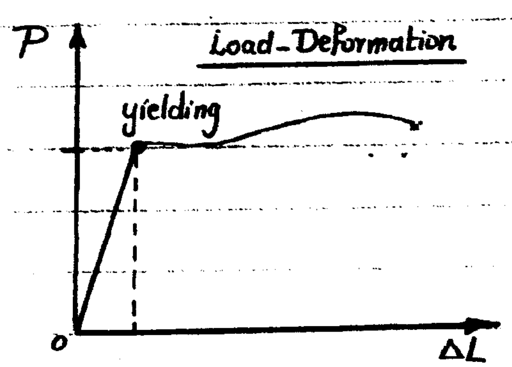


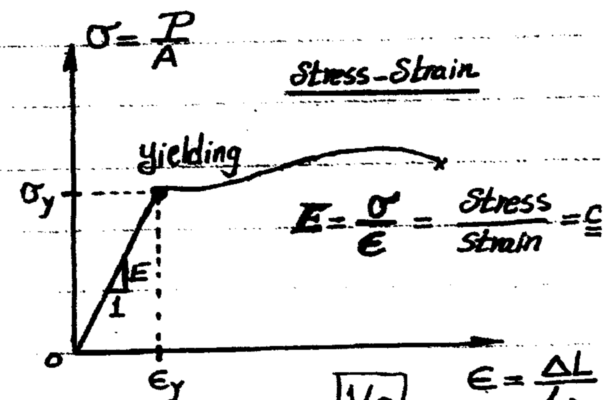
ϵ = Ratio of elongation per unit length.

$$\Rightarrow \boxed{\epsilon = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o}} = \text{Axial Strain ..... (unitless).}$$

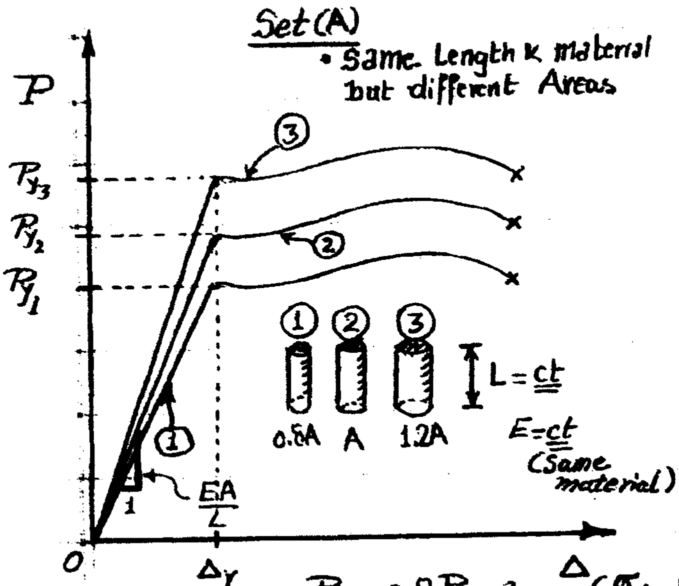
- Axial strain could be tensile or compressive (similar to stress), it is expressed in m/m, mm/mm, in./in. or simply without units. It may also be expressed as % ; i.e., $\epsilon = 0.002 \Rightarrow 0.2\%$ strain.
- How strain is measured?

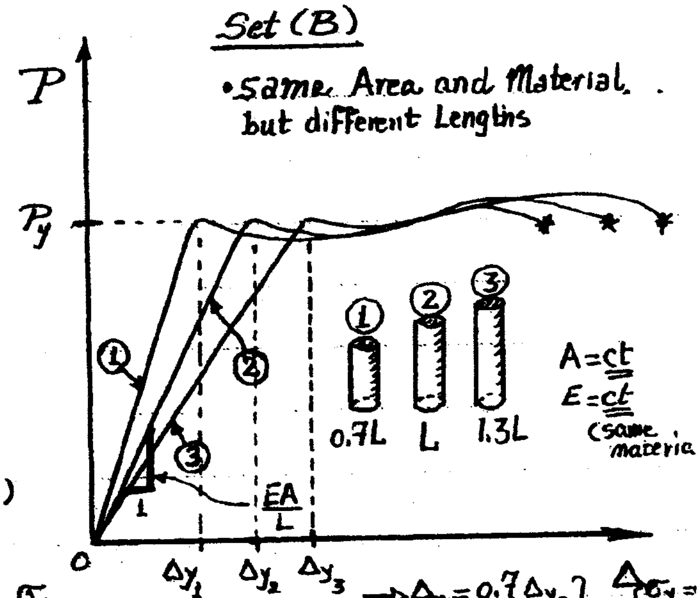
  Strain is more representative than deformation, however it is usually very small and difficult to measure manually. Special electronic devices are made to measure axial strain, such as strain gages or extensometers.





❀ Load_Deformation Curve ($P-\Delta$) Mild Steel

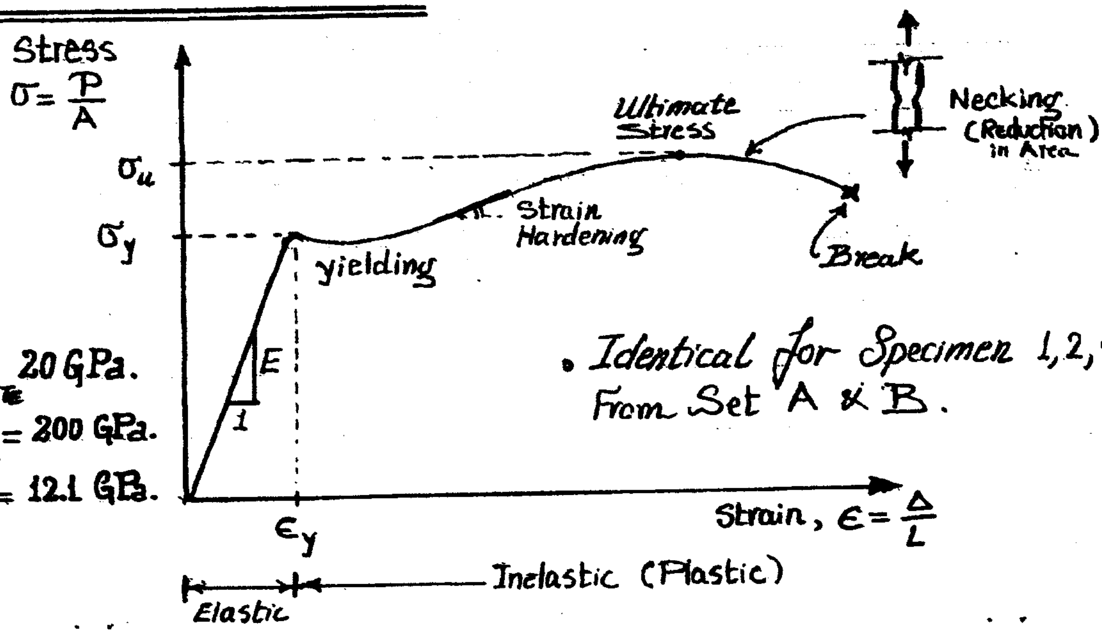




$$\Rightarrow \left.\begin{matrix} P_{y_1} = 0.8\,P_{y_2} \\ P_{y_3} = 1.2\,P_{y_2} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \sigma_{y_1} = \sigma_{y_2} \\ \sigma_{y_3} = \sigma_{y_2} \end{matrix}\right. \Rightarrow \underline{\underline{\sigma_y}}$$

$$\Rightarrow \left.\begin{matrix} \Delta_{y_1} = 0.7\,\Delta_{y_2} \\ \Delta_{y_3} = 1.3\,\Delta_{y_2} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \sigma_{y_1} = \\ \sigma_{y_3} = \end{matrix}\right.$$

❀ Stress_Strain Curve ($\sigma - \epsilon$) Mild Steel



• Identical for Specimen 1, 2, & 3 From Set A & B.

- $E_{CONCRETE} = 20$ GPa.
- $E_{Steel} = 200$ GPa.
- $E_{WOOD} = 12.1$ GPa.

Elastic Behavior

$$\Rightarrow \sigma = \underset{\substack{\uparrow \\ \text{Slope} \\ (\sigma-\epsilon)}}{E} * \epsilon \quad \Rightarrow \quad \frac{P}{A} = E \cdot \frac{\Delta}{L} \quad \Rightarrow \quad \boxed{\Delta = \frac{PL}{EA}} \quad \text{or} \quad P = \underbrace{\frac{EA}{L}}_{\substack{\text{Slope} \\ (P-\Delta)}} * \Delta$$

Modulus Of Elasticity (Young's Modulus _ E)

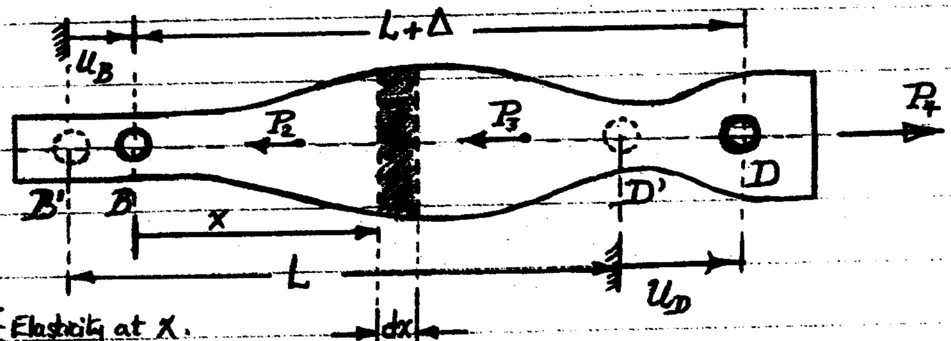
By inspection of the (σ _ ϵ) curve, the ascending initial part is linear until yielding of material.

$\Rightarrow$ for small infinitesmal deformation $\Rightarrow \dfrac{\sigma}{\epsilon} = ct = E$

E is constant for every material ... (provided in tables).

$\Rightarrow$ Modulus of Elasticity $= \boxed{E = \dfrac{\sigma}{\epsilon}}$ ... (Hook's Law _ Elastic Material).

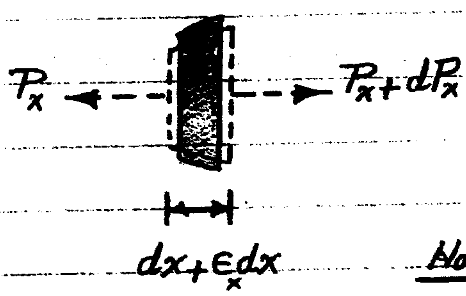
Axial Deformation



u = displacement at a section.

P_x, A_x, E_x = Force, Area, and modulus of Elasticity at x.

$$\epsilon_x = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}.$$



$$\Rightarrow \int_0^L du = \int_0^L \epsilon_x . dx.$$

$$\Rightarrow \mathcal{U}_{(L)} - \mathcal{U}_{(0)} = \int_0^L \epsilon_x . dx$$

So,

$$\Delta = \mathcal{U}_D - \mathcal{U}_B = \int_0^L \epsilon_x . dx.$$

Hook's Law: (Elastic).

$$\left.\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E_x} \\ \sigma_x &= \frac{P_x}{A_x} \end{aligned}\right\} \Rightarrow \epsilon_x = \frac{P_x}{E_x A_x}$$

$$\Rightarrow \boxed{\Delta = \int_0^L \frac{P_x . dx}{E_x . A_x}}. \quad \text{OR} \quad \boxed{\Delta = \sum_i \frac{P_i L_i}{E_i A_i}}$$

Notes

- Procedure is summarized by the 3 steps shown.
- For a homogenous and prismatic bar; i.e., $E_x = ct$ and $A_x = ct$, the strain diagram is of same degree function as N.F.D.
- From the strain diagram $\Rightarrow \Delta = \int \epsilon_x dx$; Displacement diagram is one degree higher; $\epsilon_x = \frac{du}{dx}$

$$\Rightarrow u\Big]_A^B = \int_A^B \epsilon_x . dx$$

$$\Rightarrow u_B - u_A = \text{Area under strain diagram between A \& B}$$

+ve area $\Rightarrow$ Tensile strain $\Rightarrow$ elongation

-ve area $\Rightarrow$ Compressive strain $\Rightarrow$ reduction.

(1) Normal Force Diagram (N.F.D) — $P_x = \checkmark$ Equilibrium

(2) Strain Diagram — $\epsilon_x = \frac{P_x}{E_x . A_x} = \frac{\sigma_x}{E_x}$

(3) Axial Displacement (u_x) — $\Delta = \int_0^L \frac{P_x}{E_x . A_x} dx$

Example

$P_x = P$ (Tensile Force).

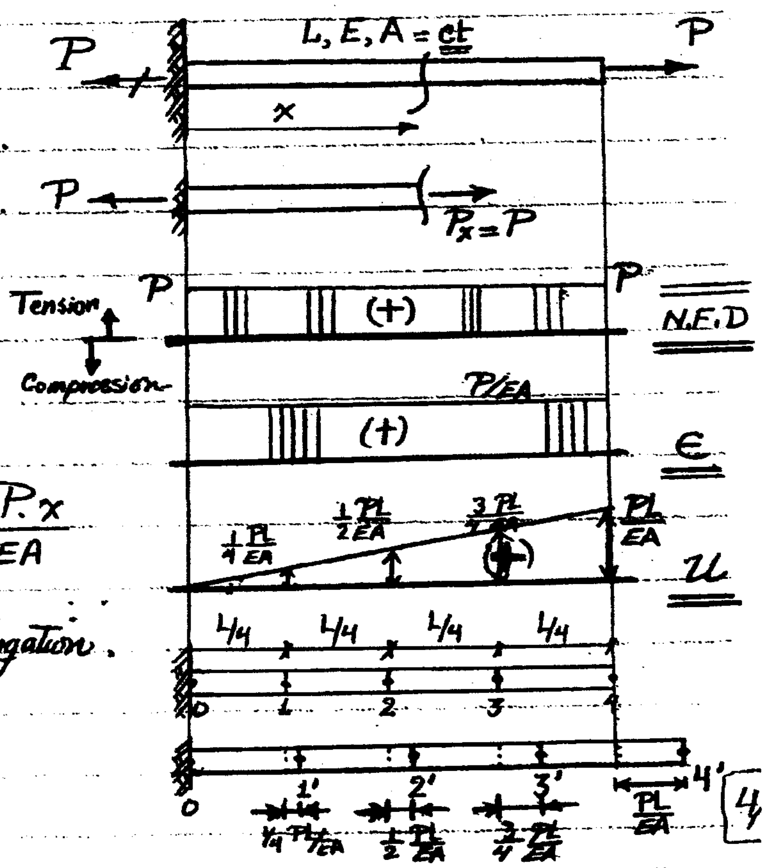
$\sigma_x = \frac{P_x}{A} = +\frac{P}{A}$ (Tension).

$\epsilon_x = \frac{\sigma_x}{E_x} = +\frac{P}{EA}$

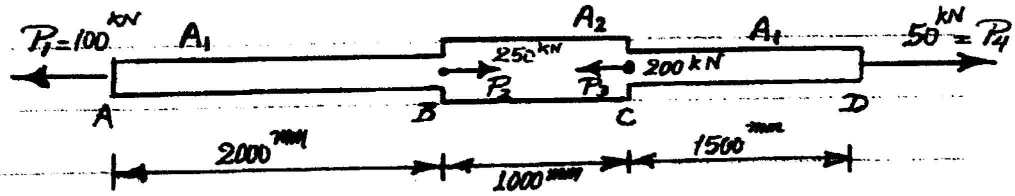
$$\Delta_x = u_x - \cancel{u_0}^{\,0} = \int_0^x \frac{P_x}{EA} dx$$

$$\Rightarrow u_x = +\frac{P}{EA} \int_0^x dx = \oplus \frac{P . x}{EA}$$

$\oplus$ indicates elongation.



Example.



$P_1 = 100^{kN}$

$P_2 = 250^{kN}$

$P_3 = 200^{kN}$

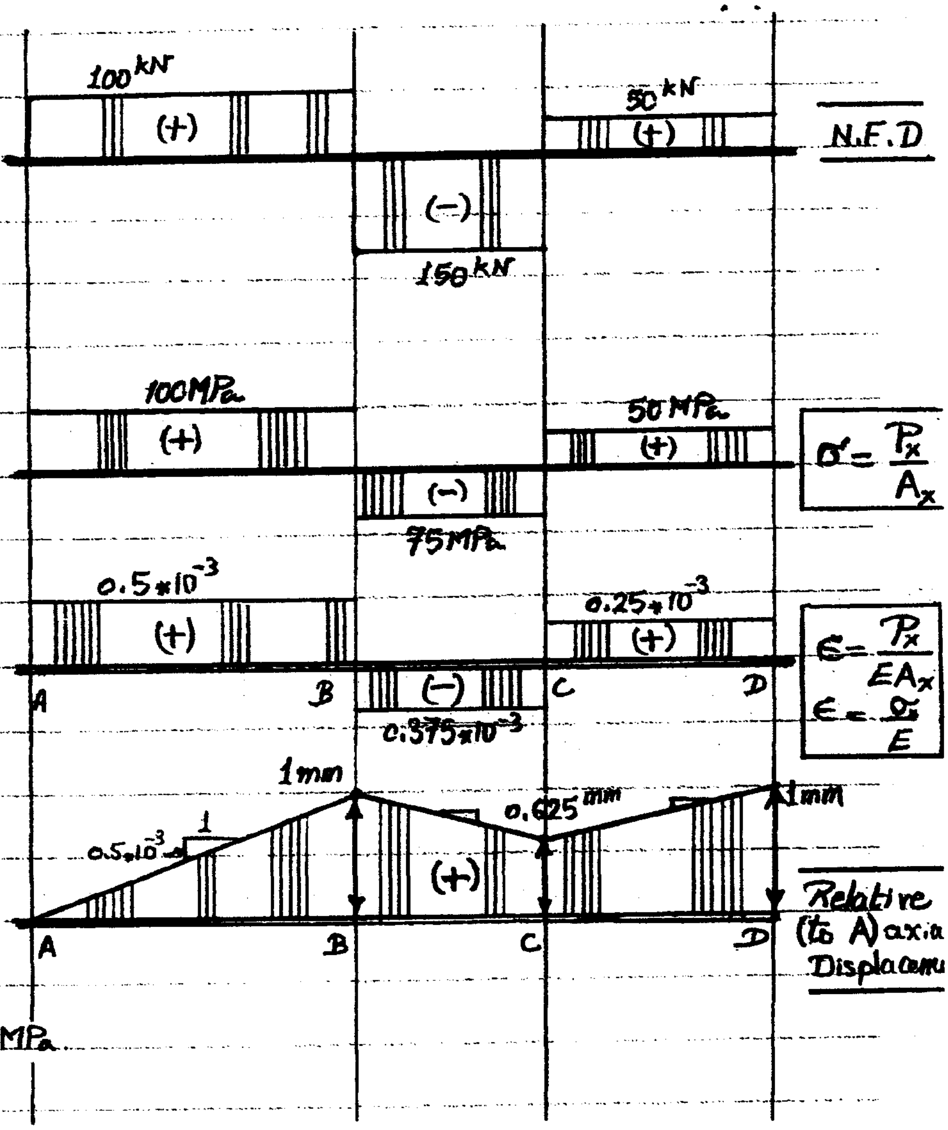
$P_4 = 50^{kN}$

$A_1 = 1000\ mm^2$

$A_2 = 2000\ mm^2$

$E = 200\ GPa$

$\Delta_{AD} = ??$



Stresses

- $\sigma_{AB} = \frac{+100^{kN}}{1000\ mm^2}$
  $= 0.1 \frac{kN}{mm^2}$
  $= 100{,}000\ kPa$
  $= 100\ MPa$

- $\sigma_{BC} = \frac{-150^{kN}}{2000\ mm^2}$
  $= -0.075 \frac{kN}{mm^2}$
  $= -75\ MPa$

- $\sigma_{CD} = \frac{+50^{kN}}{1000\ mm^2} = 50\ MPa$

Displacements

$$U_B - U_A = \int_A^B \epsilon_x\, dx = \text{Area of } \epsilon\text{-diagram between A \& B}.$$

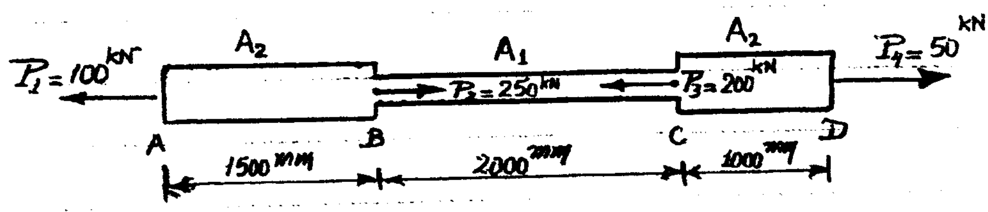
$$\Delta_{AD} = U_D - U_A = \text{Area (AB)} + \text{Area (BC)} + \text{Area (CD)}$$

$$= 10^{-3}\{0.5(2000) - 0.375(1000) + 0.25(1500)\} = (+)\,1.00\ mm.$$

Elongation.

OR $\quad \Delta_{AD} = \sum_i \frac{P_i L_i}{EA_i}$

Example



$P_1 = 100^{kN}$

$P_2 = 250^{kN}$

$P_3 = 200^{kN}$

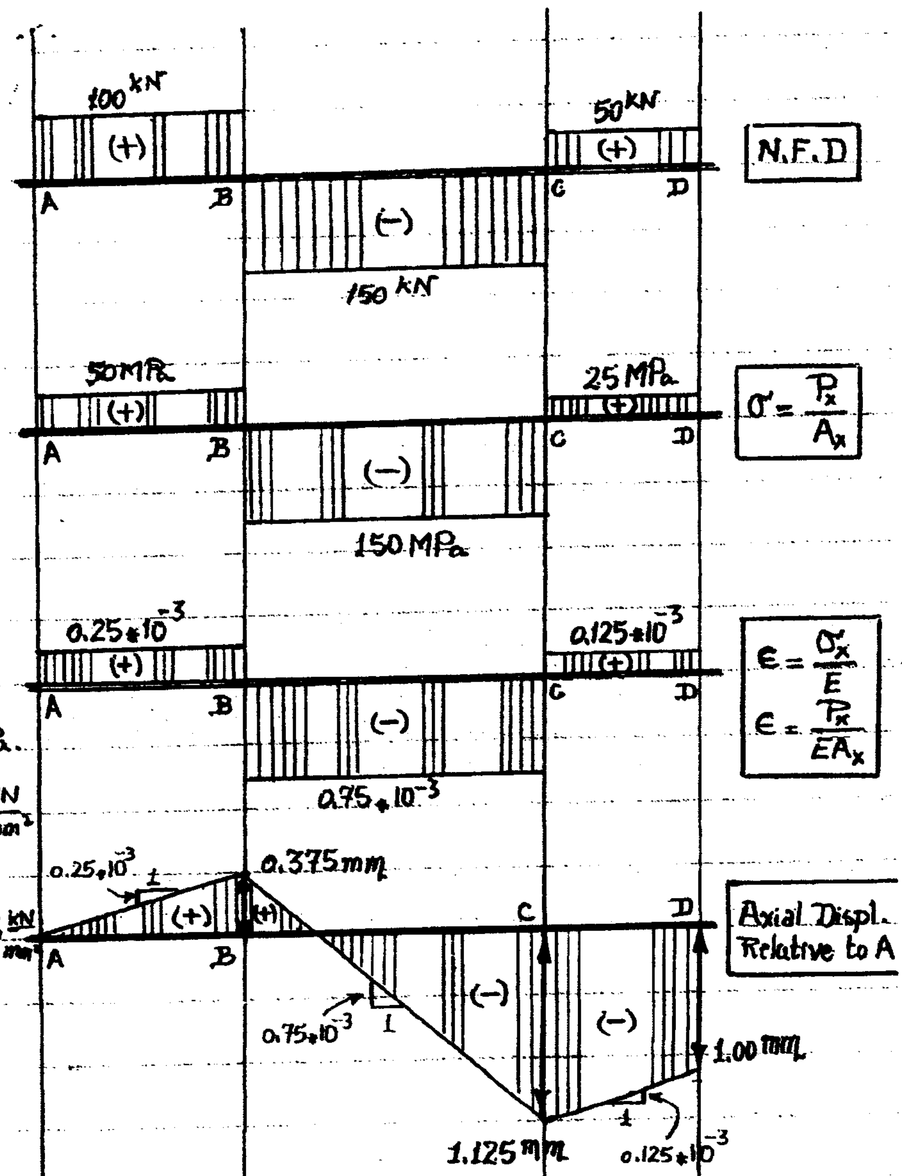
$P_4 = 50^{kN}$

$A_1 = 1000\ mm^2$

$A_2 = 2000\ mm^2$

$E = 200\ GPa$

$\Delta_{AD} = ??$



Stresses

- $\sigma_{AB} = \frac{+100^{kN}}{2000\ mm^2} = 0.05\ \frac{kN}{mm^2}$
  $= ~~50,000\ KPa~~ = 50 MPa$
- $\sigma_{BC} = \frac{-150^{kN}}{1000\ mm^2} = -0.15\ \frac{kN}{mm^2}$
  $= -150\ MPa$
- $\sigma_{CD} = \frac{+50^{kN}}{2000\ mm^2} = +0.025\ \frac{kN}{mm^2}$
  $= +25\ MPa$

Strains

$$\epsilon_x = \frac{\sigma_x}{E}$$

Displacements

$$\Delta_A^B = u_B - u_A = \int_A^B \epsilon_x\, dx = \text{Area of } \epsilon\text{-diagram between A and B.}$$

$$\Delta_A^D = u_D - u_A = \text{Area (AB)} + \text{Area (BC)} + \text{Area (CD)}.$$

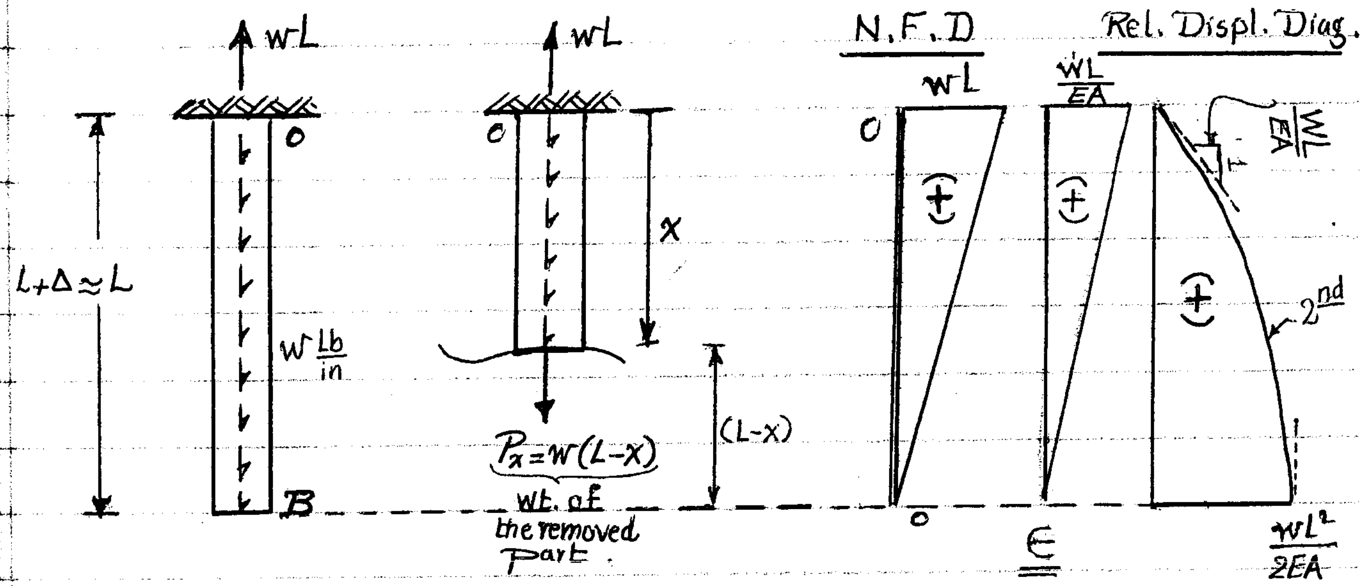
$$= 10^{-3}\,[0.25(1500) - 0.75(2000) + 0.125(1000)] = -1.000\ mm.$$

Reduction

OR $\Delta_{AD} = \sum_{i=1}^{3} \frac{P_i L_i}{E A_i}$

Example :- Variable Axial Force $P_{(x)}$.

Determine the deflection of free end B of elastic bar OB caused by its own weight w lb/in. The constant cross-sectional area is A. Assume that 'E' is constant.



$P_x = w(L-x)$ .... represents effect of the truncated segment.

$$\Delta_x = \int_0^x \frac{P_x}{EA_x}\,dx \qquad E \text{ and } A \text{ are constants.}$$

$$\Rightarrow \Delta_x = \frac{1}{EA}\int_0^x w(L-x)\,dx = \frac{1}{EA}\,w\left(Lx - \frac{x^2}{2}\right).$$

..... Eqn of 2nd degree Parabola.

at $x=0$ ... (at pt O) $\Rightarrow \Delta_o = 0$

at $x=L$ .. (at pt B) $\Rightarrow \Delta_B = u_B - \cancel{u_o}^{\,0} = \dfrac{w}{EA}\left(Lx - \dfrac{x^2}{2}\right)\Big|_0^L$

$$= \frac{w}{EA}\,\frac{L^2}{2} = \frac{wL^2}{2EA}.$$

But $wL = W$ = total wt. of the bar.

$$\Rightarrow \Delta_B = \frac{W.L}{2EA} = \frac{wL^2}{2EA}.$$

Numerical example

Assume $L = 2\,m$, circular cross section $D = 40^{cm}$, Material is Concrete.

Concrete $\Rightarrow E = 20$ GPa, $\gamma_c = 2.5$ ton/m³ ... (unit weight of concrete).

$A = \dfrac{\pi D^2}{4} = \dfrac{\pi(0.4)^2}{4} = 0.12566\,m^2$.

$w = \gamma_{conc.} * A = 2.5\,(0.12566) = 0.31416$ t/m.

OR $W = \gamma_{conc} * (A * L) = 0.6283$ tons.

$$\Rightarrow \Delta_B = \frac{0.31416 * 10^3 * 2.0^2 * 9.81}{2 * 20 * 10^9 \frac{N}{m^2} * 0.12566\,m}$$

$0.2450 * 10^{-5}\,m = 2.45 * 10^{-3}\,mm$

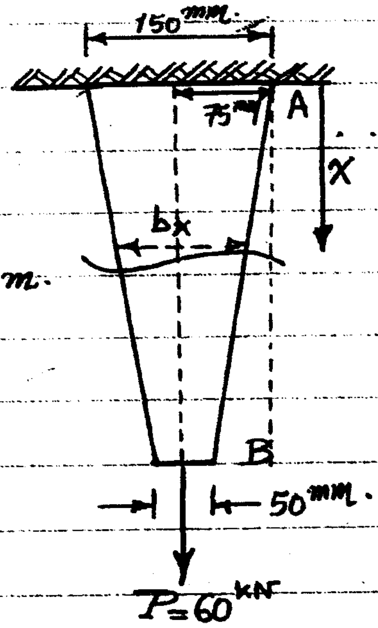
VERY SMALL.

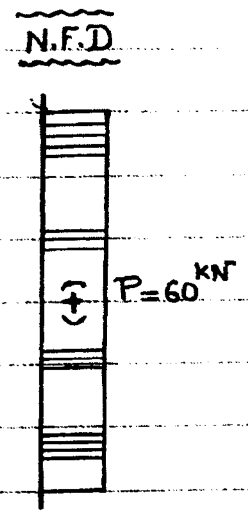
Axial Deformation in Tapered Rods

Plate thickness = 25 mm.

$E = 200\ GPa$.

Determine $\Delta_B = ?$





- Axial Force in AB = + P (SEE N.F.D).

- $\Delta_A^B = u_B - \cancel{u_A} = \int_0^L \epsilon_x \, dx$.

$$\Rightarrow u_B = \Delta_B = \int_0^L \frac{\sigma_x}{E} dx = \int_0^L \frac{P_x}{EA_x} dx \qquad P_x = +P = +60\ kN$$

$$\Rightarrow \Delta_B = \frac{P}{E}\int_0^L \frac{1}{A_x} dx \qquad A_x = t \cdot b_x = 2t\left(75 - \frac{50}{L} x\right)$$

$$b_x = 2\left(75 - \frac{50}{L} x\right) \ldots mm$$

$$\Rightarrow A_x = 50\left(75 - \frac{50x}{L}\right) \ldots mm^2$$

$$\Rightarrow \Delta_B = \frac{P}{E}\int_0^L \frac{dx}{50\left(75 - \frac{50x}{L}\right)}$$

$$= \frac{P}{50E}\int_0^L \frac{dx}{75 - \frac{50x}{L}} = \frac{P}{50E}\left[\ln\left(75 - \frac{50x}{L}\right) * \frac{1}{\left(-\frac{50}{L}\right)}\right]_0^L$$

$$= \frac{-PL}{50^2 \cdot E}\left[\ln(75 - 50) - \ln 75\right] = 4.3944 * 10^{-4} * \frac{PL}{E}$$

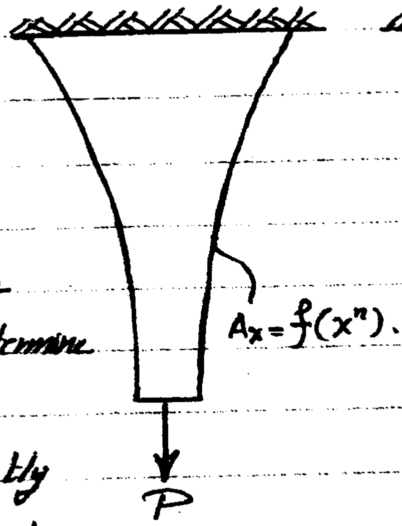
Now, having $P = 60\ kN$; $L = 3000\ mm$; $E = 200\ G\frac{N}{m^2} = 200\frac{kN}{mm^2}$

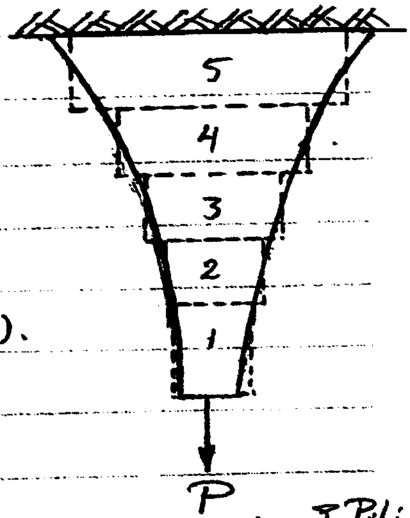
$$\Rightarrow \Delta_B = +0.3955\ mm \downarrow$$

Non-Prismatic Rods

In case of non prismatic bar, the area of the rod is variable, in such cases a proper function must be substituted for A_x, to determine deflections.

A practical and sufficiently accurate method for analysis of these problems is done by approximating the shape of the rod by a finite number of small element as shown (prismatic elements).





$$\Delta_x = \sum_i \frac{P_i L_i}{E A_i}$$